J. Fluid Mech. (1996), vol. 307, pp. 85–99

Onset of folding in plane liquid films

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(Received 14 October 1994 and in revised form 25 August 1995)

The onset of the folding effect characteristic of highly viscous liquid films (plane jets) slowly impinging on a wall is studied. Nonlinear quasi-one-dimensional equations are derived to describe the flow. In the linear approximation they reduce to the eigenvalue problem, whose solution predicts that instability (the onset of folding) sets in when the length of the film exceeds a critical value. The critical folding heights and the oscillation frequencies at the onset of instability are predicted as a function of flow parameters. Theoretical results are compared with Cruickshank's (1988) experimental data. Agreement is quite good only in the range of parameters where the quasi-one-dimensional approximation is applicable (thin films at the onset of folding).

1. Introduction

Interest in the folding phenomenon in liquid films was stimulated by an article by Taylor (1969). Detailed reviews of folding phenomena in hydrodynamics (as well as of the related buckling phenomena) have been published by Entov & Yarin (1984*b*) and Bejan (1987). To put the present work in context we consider briefly some of the references reviewed by these authors, as well as additional ones.

Buckmaster (1973) studied theoretically buckling of a thin viscous jet slowly falling in a state of near-neutral buoyancy through another liquid. A small constant drag acting on the jet from the bath liquid also was accounted for, whereas surface tension was neglected. Buckmaster, Nachman & Ting (1975) considered theoretically buckling of a thin liquid thread (viscida) as its ends are slowly drawn together. Buckmaster & Nachman (1978) incorporated in the latter problem the effect of surface tension.

The works of this group use an approach which may be characterized as 'implicitly quasi-one-dimensional'. They treat the case where the basic unperturbed liquid thread is uniform (in viscida the basic state is uniform and time-dependent), which precludes application of the results to the case where the basic flow is non-uniform at order one. The folding of thin liquid films (or buckling of jets) flowing from a slit (orifice) and impinging upon a plate studied experimentally by Cruickshank & Munson (1981) and by Cruickshank (1988) provided such an example. Griffiths & Turner (1988) conducted experiments on plane liquid films impinging on the free surface of another immiscible liquid, or on an interface between two immiscible liquids – cases where compression is of order one. Such situations with strong compression of a film or a jet are of interest in the production of non-woven materials, or in geophysical applications. In the latter, buckling of a thin liquid layer in shear is also of importance (see Benjamin & Mullin 1988 and references therein).

The folding of thin liquid films impinging upon a plate (or buckling of thin liquid jets), studied by Cruickshank & Munson (1981) and Cruickshank (1988) attracted attention as a characteristic example of low-Reynolds-number instability. Folding of

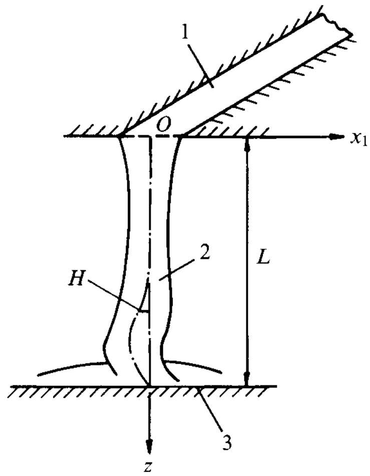

FIGURE 1. Folding of a film (plane jet) impinging upon a plate. 1, slit; 2, vertical cross-section of film; 3, plate. Dashed-dotted lines represent the unperturbed and perturbed film axes.

plane jets sets in only if the Reynolds number *Re* (under the conditions of film issue at the slit) does not exceed 0.56 (Cruickshank & Munson 1981). For $Re < 0.56$ the film length should exceed some critical value to cause folding. For Reynolds numbers larger than 0.56 the films are rectilinear and stable irrespective of their length.

Folding of impinging liquid films is characteristic of such technological processes as production of non-woven materials, coating, and rapid solidification, as well as of such geophysical processes as lithospheric slab instability in the Earth's mantle (the latter is discussed in Griffiths & Turner 1988).

Cruickshank (1988) proposed a semi-empirical theory for the folding phenomenon of plane films, as well as for buckling of axisymmetric jets (both impinging upon a plate). In this theory surface tension and gravity force are neglected. Tchavdarov, Yarin & Radev (1993) developed a theory for buckling of an axisymmetric liquid jet in the framework of the quasi-one-dimensional approach to the theory of liquid jets (Entov & Yarin 1984*a* and Yarin 1993). Their results, accounting for all the forces involved, agree fairly well with experimental data, even though no empirical parameters are used.

A similar quasi-one-dimensional approach is adopted in the present work. In §2 the quasi-one-dimensional equations of thin liquid films (plane jets) are obtained. Then in §3 this system of equations is linearized and reduced to an eigenvalue problem, which is solved numerically. The results obtained are presented, discussed and compared with the experimental data of Cruickshank (1988) in §4. The results obtained in the paper are summarized in §5.

2. Governing equations

Asymptotic quasi-one-dimensional equations of liquid motion in non-steady curved axisymmetric jets were derived by Entov & Yarin (1984*a*). In the present section we employ a similar approach to arrive at the equations for thin plane jets of films (figure 1).

Consider a smooth time-dependent two-dimensional film axis $\Gamma(t)$, given parametrically by the equation $\boldsymbol{r} = \boldsymbol{R}(s, t)$, $s_- \leqslant s \leqslant s_+$, where s is an arbitrary parameter and t time. A cross-section of a plane jet is normal to Γ and the jet (film) is considered thin provided

$$\epsilon = \max\left(\frac{a}{l}, ka\right) \ll 1, \tag{2.1}$$

where $h = 2a$ is the film thickness, l the characteristic lengthscale along the film axis and k its curvature. The position of a point or material particle in the film is determined by two parameters, s and y, where y is the coordinate in the cross-section along the normal to the film axis. The radius vector of that point or material particle is defined as

$$\boldsymbol{r}(y,s,t) = \boldsymbol{R}(s,t) + y\boldsymbol{n}(s,t), \tag{2.2}$$

where $\boldsymbol{n}$ is the unit normal vector.

At each point of the plane jet two velocities are defined: the velocity $\boldsymbol{u}$ of motion of a point with fixed coordinates (y, s) (the reference-frame velocity associated with the jet) and the velocity $\boldsymbol{v}$ of motion of a material particle (the absolute velocity):

$$\boldsymbol{u} = \frac{\partial \boldsymbol{r}(y,s,t)}{\partial t}, \quad \boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}(y,s,t)}{\mathrm{d}t}, \tag{2.3a,b}$$

$$\boldsymbol{U} = \boldsymbol{u}(0,s,t) = \frac{\partial \boldsymbol{R}}{\partial t}, \quad \boldsymbol{V} = \boldsymbol{v}(0,s,t) = \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t}. \tag{2.3c,d}$$

Here $d(\;)/\mathrm{d}t$ denotes material or substantial time differentiation in the ordinary hydrodynamic sense following the motion of the fluid (Lamb 1932, pp. 3 and 4), and $\boldsymbol{U}$ is the reference-frame velocity of the centre of the cross-section, whereas $\boldsymbol{V}$ is the velocity of the liquid at that centre.

The mass of liquid in the film between two cross-sections s_1 and s_2 is

$$M = \int_{s_1}^{s_2}\left[\int_{-a}^{a} \rho\lambda(1-ky)\,\mathrm{d}y\right]\mathrm{d}s, \quad \lambda = \left|\frac{\partial \boldsymbol{R}}{\partial s}\right|, \tag{2.4a,b}$$

where ρ is density and λ is the axial stretch.

The film cross-section with a fixed value of s moves. Therefore, transfer of mass (as well as momentum and moment of momentum) through it occurs at a velocity $(\boldsymbol{v}-\boldsymbol{u})\cdot\boldsymbol{\tau} = v_\tau - u_\tau$, where $\boldsymbol{\tau}$ is the unit tangent vector of the film axis. The mass flux through the film cross-section is

$$Q_m = \int_{-a}^{a} \rho(v_\tau - u_\tau)\,\mathrm{d}y. \tag{2.5}$$

The rate of change in the liquid mass enclosed between the cross-sections s_1 and s_2 is equal to the difference of the mass fluxes through these cross-sections, which yields as $s_2 \rightarrow s_1$ the differential continuity equation in the form

$$\frac{\partial \rho\lambda h}{\partial t} + \frac{\partial}{\partial s}\left[\rho\int_{-a}^{a}(v_\tau - u_\tau)\,\mathrm{d}y\right] = 0. \tag{2.6}$$

Here and hereinafter the indexes τ and n correspond to the tangent and normal of the film axis.

Below we consider an incompressible liquid.

The folding process itself is a low-Reynolds-number phenomenon (Cruickshank & Munson 1981). Therefore, in the momentum and moment of momentum balance we neglect the inertial forces and their moments (the corresponding estimate is given below in §4). We also neglect the forces and moments of forces imposed on the film by a gas environment. As a result we arrive at the following equations:

$$\frac{\partial}{\partial s}\left[\int_{-a}^{a} \boldsymbol{\sigma}_\tau \,\mathrm{d}y\right] + \rho\lambda h \boldsymbol{g} = 0, \tag{2.7a}$$

$$\frac{\partial}{\partial s}\left[\int_{-a}^{a} \boldsymbol{r}\times\boldsymbol{\sigma}_\tau \,\mathrm{d}y\right] + \rho\lambda h \boldsymbol{R}\times\boldsymbol{g} - \left[\rho\lambda k\int_{-a}^{a} y^2\boldsymbol{n}\,\mathrm{d}y\right]\times\boldsymbol{g} = 0, \tag{2.7b}$$

where $\boldsymbol{\sigma}_\tau(y,s,t)$ denotes the vector of stresses acting in the cross-section ($\boldsymbol{\sigma}_\tau = \boldsymbol{\tau}\cdot\boldsymbol{\sigma}^*$, where $\boldsymbol{\sigma}^*$ is the stress tensor) and $\boldsymbol{g}$ the acceleration due to gravity

Forming the vector product of $\boldsymbol{R}(s,t)$ with (2.7*a*) and subtracting it from (2.7*b*), we arrive at

$$\frac{\partial}{\partial s}\left[\int_{-a}^{a} y\boldsymbol{n}\times\boldsymbol{\sigma}_\tau \,\mathrm{d}y\right] + \lambda\boldsymbol{\tau}\times\int_{-a}^{a}\boldsymbol{\sigma}_\tau\,\mathrm{d}y - \rho\lambda k\left[\int_{-a}^{a} y^2\boldsymbol{n}\,\mathrm{d}y\right]\times\boldsymbol{g} = 0. \tag{2.8}$$

Retaining in (2.6), (2.7*a*) and (2.8) the main terms in the thin-film approximation under the conditions (2.1), we arrive at the following asymptotic, quasi-one-dimensional equations of continuity, momentum and moment of momentum:

$$\frac{\partial\lambda h}{\partial t} + \frac{\partial}{\partial s}[(V_\tau - U_\tau)h] = 0,\quad \frac{1}{\rho}\frac{\partial}{\partial s}[F\boldsymbol{\tau}+\boldsymbol{Q}] + \lambda h\boldsymbol{g} = 0,\quad \frac{1}{\rho}\left(\frac{\partial\boldsymbol{M}}{\partial s} + \lambda\boldsymbol{\tau}\times\boldsymbol{Q}\right) - \lambda k I\boldsymbol{n}\times\boldsymbol{g} = 0, \tag{2.9a–c}$$

$$F = \boldsymbol{\tau}\cdot\int_{-a}^{a}\boldsymbol{\sigma}_\tau\,\mathrm{d}y,\quad \boldsymbol{Q} = \int_{-a}^{a}\boldsymbol{\sigma}_\tau\,\mathrm{d}y - F\boldsymbol{\tau},\quad \boldsymbol{M} = \int_{-a}^{a} y\boldsymbol{n}\times\boldsymbol{\sigma}_\tau\,\mathrm{d}y,\quad I = \int_{-a}^{a} y^2\,\mathrm{d}y = \frac{2a^3}{3}. \tag{2.9d–g}$$

The quantities $\boldsymbol{\tau}F$, $\boldsymbol{Q}$, $\boldsymbol{M}$ and I represent the longitudinal force, the shearing force, the moment of stresses in the film cross-section, and the moment of inertia of the film cross-section.

In the inertialess approximation the quasi-one-dimensional equations (2.9) are valid for large deflections of the film provided inequality (2.1) is not violated. Equations (2.9*b*, *c*) are well known in the theory of rod bending (cf. Landau & Lifshitz 1959, (19.2) on p. 79 and (19.3) on p. 80). They inevitably appear in low-dimensional models of liquid jets/films (or elastic rods). For example, equations (2.9*b*, *c*) with $\boldsymbol{g} = \boldsymbol{0}$ are identical with equations (2.6*a–c*) of Buckmaster *et al.* (1975) who used them to describe buckling of a thin liquid thread as its ends are slowly drawn together. In general, considering a jet/film as a Cosserat line, one always arrives at similar quasi-one-dimensional equations. The only difference is in the terms accounting for radial inertial effects in a cross-section, which are small for long-wave perturbations (e.g. see Green 1976; Entov & Yarin 1984*b*, p. 125; Yarin 1993, p. 31). Note also that in the present work all inertial effects are neglected.

As in Entov & Yarin (1984*a*) and Yarin (1993), we adopt an analogue of the hypothesis of flat cross-sections in the theory of rod bending and suppose that at each instant the velocity $\boldsymbol{v}$ of liquid particles in the film cross-section reduces mainly to a combination of the translational motion with the centre, rigid-body rotation about it, and expansion or contraction in the direction of the normal to the film axis:

$$\boldsymbol{v} = \boldsymbol{V} + \Omega_b\boldsymbol{b}\times y\boldsymbol{n} + \delta y\boldsymbol{n} + \boldsymbol{v}_2 = \boldsymbol{V} - \Omega_b y\boldsymbol{\tau} + \delta y\boldsymbol{n} + \boldsymbol{v}_2 \tag{2.10a}$$

$$\boldsymbol{V} = \boldsymbol{n}V_n + \boldsymbol{\tau}V_\tau,\quad \boldsymbol{v}_2 = \phi_1(y,s)\boldsymbol{n} + \phi_3(y,s)\boldsymbol{\tau}. \tag{2.10b, c}$$

Here $\Omega_b \boldsymbol{b}$ is the angular velocity of the material cross-section, $\boldsymbol{b}$ the unit binormal vector to the film axis, and δ the rate of expansion/contraction of the cross-section. The functions ϕ_i may be expanded in series in y, starting with the second-order terms.

The gradient operator $\boldsymbol{\nabla}$ is

$$\boldsymbol{\nabla} = \boldsymbol{n}\frac{\partial}{\partial y}+\boldsymbol{b}\frac{\partial}{\partial x}+\lambda^{-1}(1-ky)^{-1}\boldsymbol{\tau}\frac{\partial}{\partial s}, \tag{2.11}$$

where x is the coordinate along the binormal to the film axis. In a plane jet all the parameters are independent of x.

By means of (2.10) and (2.11) we obtain the components of the strain-rate tensor $\boldsymbol{D}$:

$$D_{nn} = \delta+\phi_{1,y}, \quad D_{nb} = D_{bn} = 0, \tag{2.12a,b}$$

$$D_{n\tau} = D_{\tau n} = \tfrac{1}{2}(-\Omega_b+\lambda^{-1} V_{n,s}+kV_\tau+\phi_{3,y}-yk\Omega_b+y\lambda^{-1}\delta_{,s}+yk\lambda^{-1} V_{n,s}+yk^2 V_\tau), \tag{2.12c}$$

$$D_{bb} = 0, \quad D_{b\tau} = D_{\tau b} = 0, \tag{2.12d,e}$$

$$D_{\tau\tau} = \lambda^{-1} V_{\tau,s}-kV_n-y\lambda^{-1}\Omega_{b,s}-\delta yk+yk\lambda^{-1} V_{\tau,s}-yk^2 V_n. \tag{2.12f}$$

It is emphasized that higher-order terms in y should not be retained in the thin-film approximation.

The incompressibility condition $\operatorname{tr}\boldsymbol{D} = 0$ yields

$$\delta = -\lambda^{-1} V_{\tau,s}+kV_n, \quad \phi_{1,y} = y(\lambda^{-1}\Omega_{b,s}+\delta k-k\lambda^{-1} V_{\tau,s}+k^2 V_n). \tag{2.13a,b}$$

Note that equations (2.13 a,b) result from a perturbation calculation that equates terms at each order to zero.

For a viscous Newtonian fluid

$$\boldsymbol{\sigma}^* = -p\boldsymbol{I}+2\mu\boldsymbol{D}, \tag{2.14}$$

where $\boldsymbol{\sigma}^*$ is the stress tensor, $\boldsymbol{I}$ the tensor unit, p pressure and μ viscosity.

Substituting (2.12) in (2.14) we arrive at

$$\sigma_{nn} = -p+2\mu(\delta+\phi_{1,y}), \quad \sigma_{nb} = \sigma_{bn} = 0, \tag{2.15a,b}$$

$$\sigma_{n\tau} = \sigma_{\tau n} = \mu[-\Omega_b+\lambda^{-1} V_{n,s}+kV_\tau+y(\lambda^{-1}\delta_{,s}-k\Omega_b+k\lambda^{-1} V_{n,s}+k^2 V_\tau)+\phi_{3,y}], \tag{2.15c}$$

$$\sigma_{bb} = -p, \quad \sigma_{b\tau} = \sigma_{\tau b} = 0, \tag{2.15d,e}$$

$$\sigma_{\tau\tau} = -p+2\mu[\lambda^{-1} V_{\tau,s}-kV_n+y(-\lambda^{-1}\Omega_{b,s}-\delta k+k\lambda^{-1} V_{\tau,s}-k^2 V_n)]. \tag{2.15f}$$

With surface tension taken into account, we find from the Laplace formula that the capillary pressures under and over the free surfaces at $y = \pm a$ are

$$p_\gamma^\pm = -\frac{\gamma\lambda^{-1}[\lambda^{-1}a_{,s}]_{,s}}{(1+\lambda^{-2}a_{,s}^2)^{3/2}}\mp\frac{\gamma k}{(1+\lambda^{-2}a_{,s}^2)^{3/2}}, \tag{2.16}$$

where γ is the surface tension coefficient.

Applying the conditions for the normal stress as in (2.15 a), $\sigma_{nn} = -p_\gamma^-$ at $y = -a$ and $\sigma_{nn} = -p_\gamma^+$ at $y = a$, in the thin-film approximation we arrive at the following expression for the pressure:

$$p = -\frac{\gamma\lambda^{-1}[\lambda^{-1}a_{,s}]_{,s}}{(1+\lambda^{-2}a_{,s}^2)^{3/2}}-\frac{\gamma k y}{a(1+\lambda^{-2}a_{,s}^2)^{3/2}}+2\mu(\delta+\phi_{1,y}). \tag{2.17}$$

From (2.13), (2.15f) and (2.17) we find

$$\sigma_{\tau\tau} = \frac{\gamma\lambda^{-1}[\lambda^{-1}a_{,s}]_{,s}}{(1+\lambda^{-2}a_{,s}^2)^{3/2}} + \frac{\gamma k y}{a(1+\lambda^{-2}a_{,s}^2)^{3/2}} + 4\mu(\lambda^{-1} V_{\tau,s} - kV_n)$$
$$+4\mu y(-\lambda^{-1}\Omega_{b,s} + 2\lambda^{-1} V_{\tau,s} k - 2k^2 V_n). \quad (2.18)$$

The expressions for the axial value of the longitudinal stress and longitudinal force by (2.18) are of the form

$$\Sigma_{\tau\tau} = \frac{\gamma\lambda^{-1}[\lambda^{-1}a_{,s}]_{,s}}{(1+\lambda^{-2}a_{,s}^2)^{3/2}} + 4\mu(\lambda^{-1} V_{\tau,s} - kV_n), \quad F = \Sigma_{\tau\tau} h + \frac{2\gamma}{(1+\lambda^{-2}a_{,s}^2)^{1/2}}. \quad (2.19a, b)$$

Owing to (2.9f) and (2.15e) $\boldsymbol{M} = -\boldsymbol{b}\int_{-a}^{a} y\sigma_{\tau\tau}\,\mathrm{d}y$. Therefore,

$$M_n = M_\tau = 0, \quad (2.20a)$$

$$M_b = 4\mu I(\lambda^{-1}\Omega_{b,s} - 2\lambda^{-1} V_{\tau,s} k + 2k^2 V_n) - \frac{\gamma k I}{a(1+\lambda^{-2}a_{,s}^2)^{3/2}}. \quad (2.20b)$$

The condition of smallness of stresses on the film surface (in contact with a gas environment) means that at the leading order (order one) stresses $\sigma_{n\tau}$ and $\sigma_{\tau n}$ should be zero (or more accurately, $O(\epsilon\sigma_{\tau\tau})$), which from (2.15$c$) yields

$$\Omega_b = \lambda^{-1} V_{n,s} + kV_\tau. \quad (2.21)$$

The terms linear in y in (2.15c) for $\sigma_{\tau n}$ do not contribute to the value of the shearing force $|Q|$. Therefore, in view of (2.21), $|Q|$ should be of $O(\epsilon^2 F)$. In these circumstances an explicit expression for the shearing force is unobtainable within the framework of the asymptotics of the given accuracy, and it can be only determined from the solution of the problem.

The equations of continuity, momentum and moment of momentum (2.9a–c), may be rearranged in the following form:

$$\frac{\partial\lambda h}{\partial t} + \frac{\partial}{\partial s}[(V_\tau - U_\tau)h] = 0, \quad h = 2a, \quad (2.22a, b)$$

$$\frac{1}{\rho}\left(\frac{\partial F}{\partial s} - Q_n\lambda k\right) + \lambda h g_\tau = 0, \quad \frac{1}{\rho}\left(F\lambda k + \frac{\partial Q_n}{\partial s}\right) + \lambda h g_n = 0, \quad \frac{1}{\rho}\left(\frac{\partial M_b}{\partial s} + \lambda Q_n\right) + \lambda k I g_\tau = 0. \quad (2.22c\text{–}e)$$

where g_τ and g_n denote the projections of gravity acceleration on the directions tangent and normal to the film axis.

In the projection of the momentum equation onto the tangent, (2.22c), we neglected Q_τ compared to F, since $|Q| = O(\epsilon^2 F)$, whereas in its projection onto the normal, (2.22d), $F\lambda k$ and Q_n are of the same order of magnitude owing to the smallness of curvature of the film axis in the given problem.

Note that to derive the quasi-one-dimensional equations of film folding (2.19)–(2.22) we used the integral balance method supplemented by the asymptotic representation (2.10) of the velocity field in a thin film. A similar approach has been used previously to derive quasi-one-dimensional equations of liquid fibres (Kase & Matsuo 1965; Matovich & Pearson 1969), bending and buckling liquid jets (Entov & Yarin 1984a, Yarin 1993), hollow liquid films (Taylor 1959; Pearson & Petrie 1970*a,b*; Yarin, Gospodinov & Roussinov 1994), etc. For flows with gradual longitudinal variation of parameters such a simple procedure provides exactly the same results as a direct perturbation expansion of the flow field and the Navier–Stokes equations, or averaging of the latter over a liquid cross-section (which was proved by Schultz & Davis 1982 for fibres, and by Yarin 1983 for bending and buckling jets).

Equations (2.22), supplemented by (2.19)–(2.21) and the following geometric and kinematic relations:

$$\lambda = (1+H_{,s}^2)^{1/2}, \quad k = H_{,ss}/\lambda^3, \tag{2.23a,b}$$

$$\frac{\partial H}{\partial t} = \lambda V_n, \quad U_\tau = V_n H_{,s} \tag{2.23c,d}$$

form the closed system of equations of the problem in the inertialess approximation. Note that in (2.23) we assume that $H = H(s,t)$ is the amplitude of displacement of the film axis in the direction Ox_1 in a Cartesian coordinate system $Ox_1 z$ (see figure 1), where the film axis at any moment and at all points forms an acute angle with the direction Oz of the unperturbed film axis (overturnings are forbidden since we proceed below to consider small perturbations).

It is emphasized that where required (for example, if the folding amplitude becomes so large at the nonlinear stage as to lead to overturnings), parameter s of the film axis may be taken as a Lagrangian one. Formulae (2.23) should then be modified as was done in the corresponding problem on bending instability leading to overturns when a thin highly viscous liquid jet moves in air at high speed (see Yarin 1993, pp. 105 and 106). Also in the case of drawing a film with a rectilinear axis through a slit (a counterpart of fibre spinning) or stretching jets produced by shaped charges, it is sometimes convenient to choose s as a Lagrangian parameter, namely the time moment at which a liquid particle left the slit (cf. Yarin 1993, pp. 170–172; Yarin 1994). In all these cases, all equations in the present section except (2.23), hold.

3. Eigenvalue problem

Below we use for s the Cartesian coordinate z, taken along the axis of an unperturbed film, as we are concerned with small folding perturbations. Therefore, from now on $s = z$. First, we consider the unperturbed distributions of the half-width $a^0(z)$, which correspond to rectilinear films. With longitudinal force taken as $F = 4\mu h\, \partial w/\partial z + 2\gamma$ ($w = V_\tau$), which simplifies (2.19), the continuity and longitudinal momentum equations (2.22*a,c*) reduce to

$$a^0 \frac{d^2 a^0}{dz^2} - \left(\frac{da^0}{dz}\right)^2 = \tfrac{3}{2}\beta^2 a^{0^3}, \quad \beta^2 = L_*^2 G; \quad w^0 = 1/a^0. \tag{3.1a,b}$$

Here and hereinafter in the equations in non-dimensional form we use as a scale of the half-width its value a_0 at the slit exit ($z = 0$); the coordinate z and H are scaled by L, the slit/plate distance; velocities $u = V_n$ and $w = V_\tau$ are scaled by w_0^0, the outflow velocity at the slit exit; time is scaled by L/w_0^0; $L_* = L/a_0$; the non-dimensional parameter G is

$$G = \frac{\rho g a_0^2}{6\mu w_0^0}, \tag{3.2}$$

where $g_\tau = g$ and $g_n = 0$.

Equation (3.1 a) is subject to the boundary conditions

$$z = 0: \quad a^0 = 1; \qquad z = 1: \quad a^0 = \infty \tag{3.3}$$

which represent a given film thickness at the slit exit and an impermeable plate.

The solution of (3.1 a) and (3.3) was found by Cruickshank (1984) and has the following form:

$$a^0(z) = \frac{2D}{3\beta^2} \frac{1}{\cosh[D^{1/2}(z-1)]-1}, \quad 0 < \beta < \frac{2}{\sqrt{3}}, \tag{3.4a}$$

$$a^0(z) = \frac{1}{(1-z)^2}, \quad \beta = \frac{2}{\sqrt{3}}, \tag{3.4b}$$

$$a^0(z) = \frac{2D_1}{3\beta^2} \frac{1}{\cos[(-D_1)^{1/2}(z-1)]-1}, \quad \beta > \frac{2}{\sqrt{3}}. \tag{3.4c}$$

In (3.4 a) D is a positive solution of the equation

$$D = \frac{3\beta^2}{2}[\cosh(D^{1/2})-1], \quad 0 < \beta < \frac{2}{\sqrt{3}}, \tag{3.5}$$

whereas in (3.4 c) D_1 is a solution of

$$D_1 = \frac{3\beta^2}{2}\{\cos[(-D_1)^{1/2}]-1\}, \quad \beta > \frac{2}{\sqrt{3}} \tag{3.6}$$

belonging to the interval $(-4\pi^2, 0)$.

If $\beta < \pi/\sqrt{3}$ the film is in compression from the very beginning and $\mathrm{d}a^0/\mathrm{d}z > 0$, which means that the effect of liquid slowdown by the plate and the corresponding compressive viscous force outweigh the gravity acceleration.

If $\beta > \pi/\sqrt{3}$ the film is in tension from the slit up to some cross-section ($\mathrm{d}a^0/\mathrm{d}z < 0$) and gravity acceleration predominates. The last portion of the film is, however, in compression and there $\mathrm{d}a^0/\mathrm{d}z > 0$, which corresponds to domination of the viscous force.

Consider now small folding perturbations of the straight axis of the film, when in the linear approximation

$$u = H_{,t}, \quad k = H_{,zz}, \quad \lambda = 1, \quad U_\tau = 0. \tag{3.7a–d}$$

Linearizing, we get from (2.19 a,b), (2.20 b), (2.21) and (2.22 a–e) the following dimensional equations:

$$\frac{\partial h}{\partial t} + \frac{\partial hw}{\partial z} = 0, \quad \frac{1}{\rho}\frac{\partial F}{\partial z} + hg = 0, \quad \frac{1}{\rho}\left(Fk + \frac{\partial Q_n}{\partial z}\right) - gh\frac{\partial H}{\partial z} = 0, \tag{3.8a–c}$$

where

$$F = 4\mu\frac{\partial w}{\partial z}h + 2\gamma\left[\frac{1}{(1+a_{,z}^2)^{1/2}} + \frac{aa_{,zz}}{(1+a_{,z}^2)^{3/2}}\right], \tag{3.9a}$$

$$Q_n = -\frac{\partial}{\partial z}\left\{4\mu I\left[\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + kw\right) - 2k\frac{\partial w}{\partial z}\right] - \frac{\gamma k I}{a(1+a_{,z}^2)^{3/2}}\right\} - kI\rho g, \tag{3.9b}$$

and g is acceleration due to gravity. Note that the term $-\lambda k Q_n/\rho$ in (2.22 c) is obviously quadratic in perturbations and, thus, neglected in (3.8 b).

Define an eigenfunction $f(z)$ and an eigenvalue Ψ of the displacement amplitude $H = f(z)\exp(\Psi t)$. Substituting (3.9 a,b) in (3.8 c) and bearing in mind (3.7 a,b), we find

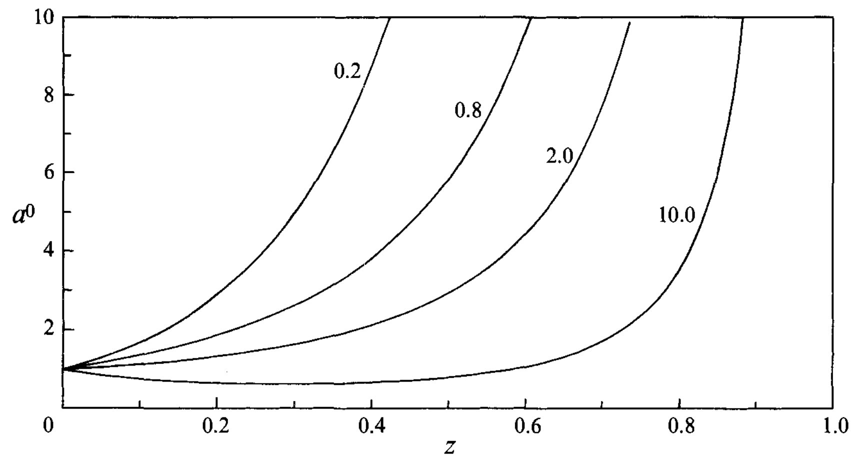

FIGURE 2. Unperturbed film profiles corresponding to values of β^2 indicated on the curves.

in the linear approximation (around the unperturbed solution (3.4)) the following expression for the eigenfunction:

$$\frac{8}{a^0}\frac{da^0}{dz}\frac{d^2f}{dz^2}-2\Gamma\left\{\frac{1}{[L_*^2+(da^0/dz)^2]^{1/2}}+\frac{a^0 d^2a^0/dz^2}{[L_*^2+(da^0/dz)^2]^{3/2}}\right\}\frac{d^2f}{dz^2}$$
$$+\frac{2}{3}\frac{1}{L_*^2}\frac{d^2}{dz^2}\left\{4a^{0^3}\left[\Psi\frac{d^2f}{dz^2}+\frac{1}{a^0}\frac{d^3f}{dz^3}+\frac{1}{a^{02}}\frac{da^0}{dz}\frac{d^2f}{dz^2}-\frac{\Gamma L_*^2}{4a^0}\frac{1}{[L_*^2+(da^0/dz)^2]^{3/2}}\frac{d^2f}{dz^2}\right]\right\}$$
$$+4GL_*^2\left[\frac{1}{L_*^2}\frac{d}{dz}\left(a^{0^3}\frac{d^2f}{dz^2}\right)+3a^0\frac{df}{dz}\right]=0, \qquad (3.10)$$

where the non-dimensional parameter Γ is

$$\Gamma=\frac{\gamma L^2}{\mu a_0^2 w_0^0}. \qquad (3.11)$$

The coefficients in (3.10) depend on the unperturbed profile of the film $a^0(z)$ which is determined by (3.4)–(3.6).

Note that the remaining equations (3.8a,b) after subtracting an unperturbed part satisfied by (3.4), yield two autonomous equations for perturbations of h and w.

Equation (3.10) is supplemented by the following boundary conditions:

$$z=0:\quad f=0,\quad df/dz=0,\quad d^2f/dz^2=0, \qquad (3.11\,a\text{–}c)$$
$$z=1:\quad f=0,\quad df/dz=0, \qquad (3.11\,d,e)$$

which express, respectively: (i) the absence of any displacement of the film axis at the slit exit ($z=0$); (ii) the smooth junction of the tangent to the film axis and the slit exit; (iii) the absence of rotation of the liquid cross-section at the slit exit (which corresponds to $\Omega_b=0$ at $z=0$); (iv) the absence of any displacement of the film axis at the plate ($z=1$); (v) clamping of the film axis at the plate.

These boundary conditions are discussed in detail by Tchavdarov *et al.* (1993). We mention here only that the clamp condition (3.11e) corresponds to infinite spreading of the film over the plate, where its thickness and moment of inertia become extremely large, arresting any motion of the film section near the plate.

Equation (3.10) is solved under the conditions (3.11) numerically by applying the method discussed in detail by Tchavdarov *et al.* (1993).

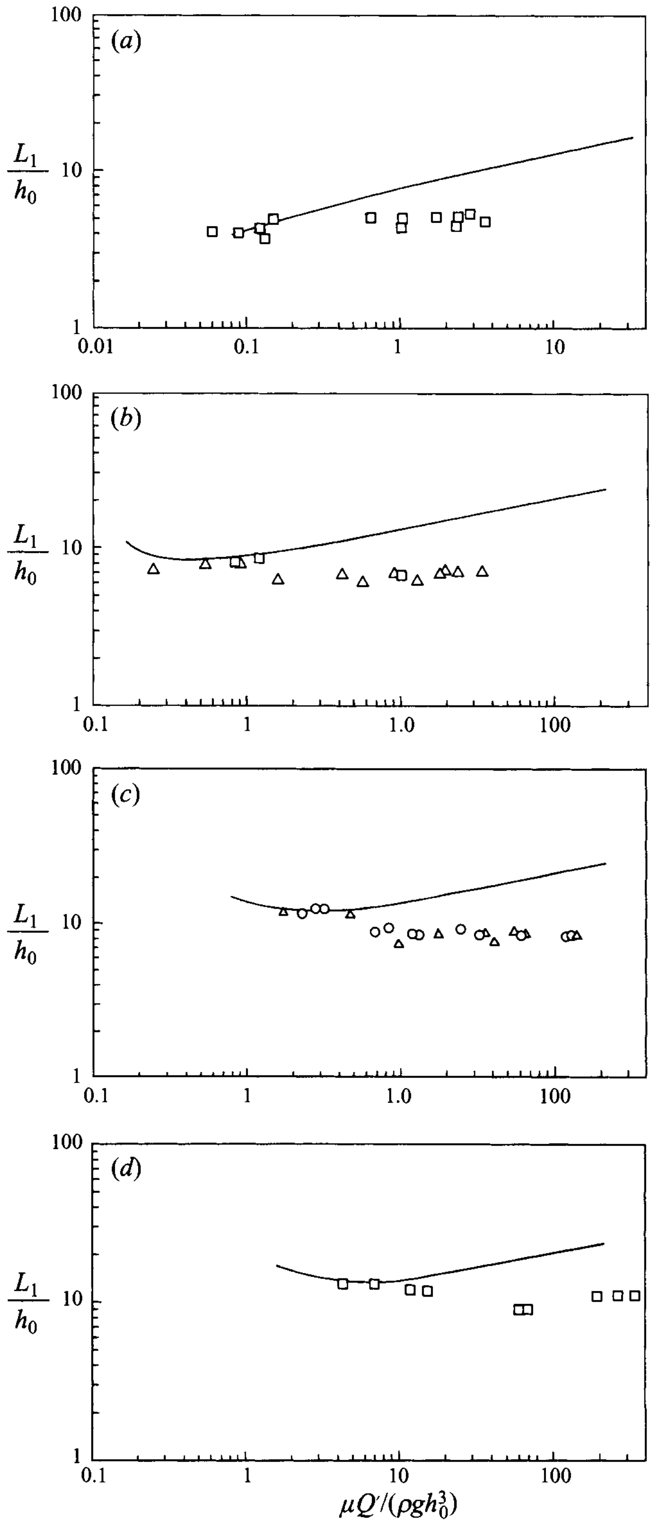

FIGURE 3. Folding height L_1 as a function of viscous, gravity and surface-tension forces. (*a*) Low surface tension: $\gamma/\rho g h_0^2 = 5.73 \times 10^{-2}$. ——, Theoretical results; □, Cruickshank's (1988) ex-

4. Results, comparison with experiment and discussion

In figure 2 the unperturbed profiles of the film are plotted for several values of β^2 as predicted by (3.4). For comparison with the experimental data of Cruickshank (1988), we note that

$$\beta^2 = \left(\frac{L}{h_0}\right)^2 \bigg/ \left(6\frac{\mu Q'}{\rho g h_0^3}\right), \tag{4.1}$$

where $Q' = w_0^0 h_0$ is the volume flow rate per unit slit width, $h_0 = 2a_0$ and w_0^0 is the outflow velocity.

Critical values of L/h_0 for plane jets according to Cruickshank (1988) are of the order of ten. Therefore, from (4.1) we find

$$\frac{\mu Q'}{\rho g h_0^3} \approx \frac{16.67}{\beta^2}. \tag{4.2}$$

The data in figure 2 show that about $\beta^2 = 0.5$ the half-width in the middle of the film is approximately equal to the half-length of the film. Therefore, only films with β^2 significantly larger 0.5 may be treated under the quasi-one-dimensional approach. According to (4.2), only films with $\mu Q'/\rho g h_0^3$ significantly smaller than 33.33 may be characterized as thin in terms of the quasi-one-dimensional theory.

The solution of the eigenvalue problem (3.10) and (3.11) predicts that instability sets in when the length of the film exceeds some critical value depending on the parameters of the flow. This manifests itself in the calculations as change of sign of the real part of the first eigenvalue from negative to positive. The results of the calculations are compared with the experimental data of Cruickshank (1988) in figure 3. It is clearly seen that the theory agrees with the experimental data only up to a value of $\mu Q'/\rho g h_0^3$ of the order of several units. For higher values the theory overestimates the folding height. This is not surprising, since such higher values do not correspond to sufficiently thin films to qualify for the quasi-one-dimensional approach.

Cruikshank & Munson (1981) and Cruickshank (1988) consider the inertial forces to be small compared with the viscous ones in their experiments with buckling and folding jets. Their ratio is given in the present case by the following Reynolds number:

$$Re_1 = \frac{Q'}{\nu}\frac{L}{h_0} = \frac{\mu Q'}{\rho g h_0^3}\frac{g h_0^3}{\nu^2}\frac{L}{h_0}. \tag{4.3}$$

Taking $\mu Q'/\rho g h_0^3 \sim 10^2$, $L/h_0 \sim 10$, $\nu = \mu/\rho \sim 10^{-2}\,\mathrm{m^2\,s^{-1}}$, $\mathrm{h}_0 \sim 10^{-3}\,\mathrm{m}$ and $g \sim 10\,\mathrm{m\,s^{-2}}$ we get $Re_1 \sim 10^{-1}$. The latter estimate allows one to neglect the inertial forces as was done in the momentum and moment of momentum equations (2.7)–(2.9). It should be added that the effect of the inertial forces increases with $\mu Q'/\rho g h_0^3$. This may partially account for the above-mentioned overestimation of the experimental data by the theoretical results at higher values of this parameter in figure 3.

The theoretical results of figure 3(*a–d*) are combined in figure 4 to illustrate the stabilizing effect of surface tension on folding instability of viscous–gravity films. Similar trends characterize buckling instability of axisymmetric liquid jets (Tchavdarov *et al.* 1993).

perimental data for $W/h_0 = 5$, where W is the slit width. (*b*) Higher surface tension: $\gamma/\rho g h_0^2 = 0.23$. ——, Theoretical results; Cruickshank's (1988) experimental data: □, $W/h_0 = 5$; △, $W/h_0 = 10$. (*c*) Still higher surface tension: $\gamma/\rho g h_0^2 = 0.59$. ——, Theoretical results; Cruickshank's (1988) experimental data for $W/h_0 = 15$; ○, $h_0 \times W = 0.24 \times 3.58$ cm; and △, $h_0 \times W = 0.198 \times 2.97$ cm. (*d*) Still higher surface tension: $\gamma/\rho g h_0^2 = 0.9$. ——, Theoretical results; □, Cruickshank's (1988) experimental data ($W/h_0 = 15$).

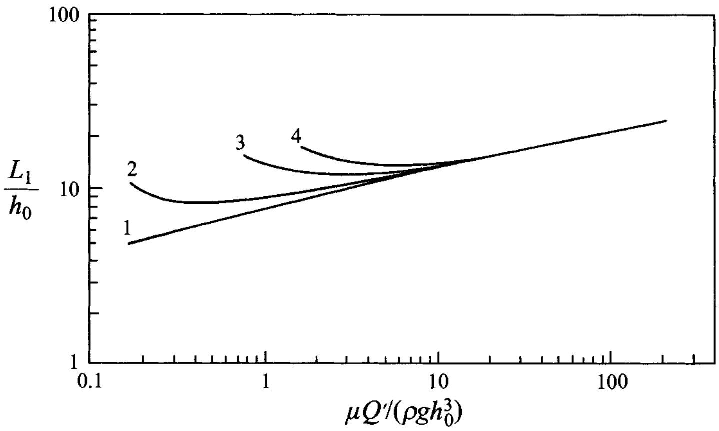

FIGURE 4. Effect of surface tension on folding height. Curves 1–4 refer to $\gamma/\rho g h_0^2 = 0$, 0.23, 0.59 and 0.9, respectively.

The reason for the existence of a critical folding height, below which the film is stable, is the following. In accordance with (2.19), (2.20*b*) and (2.22*d, e*) the balance of the following three force moments corresponds to motion of a liquid in a film:

$$M_1 = \left\{8\mu a^0 \frac{\mathrm{d}w^0}{\mathrm{d}z} + 2\gamma\left[\frac{1}{(1+a^{02}_{,z})^{1/2}} + \frac{a^0 a^0_{,zz}}{(1+a^{02}_{,z})^{3/2}}\right]\right\} H - \rho g \int_0^z \left(2a^0 H + I_0 \frac{\partial^2 H}{\partial z^2}\right) \mathrm{d}z, \tag{4.4a}$$

$$M_2 = -4\mu I_0\left(w^0 \frac{\partial k}{\partial z} - k\frac{\mathrm{d}w^0}{\mathrm{d}z}\right) + \frac{\gamma I_0 k}{a^0(1+a^{02}_{,z})^{3/2}}, \tag{4.4b}$$

$$M_3 = -4\mu I_0 \frac{\partial^3 H}{\partial z^2\, \partial t}. \tag{4.4c}$$

Formulae (4.4) are written for small folding perturbations in dimensional form; M_1 is the bending moment due to longitudinal compression by the viscous force and gravity, opposed by surface tension; M_2 is the moment of the viscous stresses due to motion of a liquid particle along a curved trajectory (also affected by surface tension); M_3 is the moment of the viscous stresses due to curvature change with time. In analogy to the case of liquid jet buckling considered by Tchavdarov *et al.* (1993), as the distance between the slit exit and the plate decreases, the moment M_2 increases sharply, since its magnitude is determined by the leading derivative $\partial k/\partial z = \partial^3 H/\partial z^3$. Therefore, for sufficiently small heights L, the bending moment M_1 cannot overcome M_2 and the flow in the film is stable. The bending moment M_1 begins to dominate only for heights above a critical one corresponding to given flow parameters.

Figure 4 shows the pattern of the critical folding height at smaller flow rates as surface tension increases. According to (4.4*a*) this means that an increase in surface tension, in the main, reduces the bending moment M_1, which leads to increase of the critical height. At higher flow rates compressive viscous force dominates the moment M_1, whereas surface tension cannot practically compete with it. As a result, at higher flow rates the critical folding height becomes practically independent of surface tension. Similar trends have been found at the onset of buckling of liquid jets (Tchavdarov *et al.* 1993).

Figures 3 and 4 show that the critical height L_1/h_0 is nearly independent of the flow

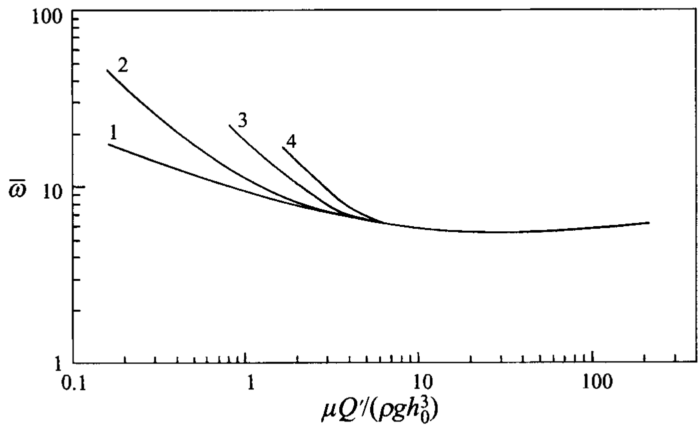

FIGURE 5. Effect of surface tension on folding frequency. Curves 1–4 refer to $\gamma/\rho g h_0^2 = 0$, 0.23, 0.59 and 0.9, respectively.

rate. This results from the fact that the non-dimensional viscous compressive force $8a^0\,\mathrm{d}w^0/\mathrm{d}z$ is only slightly dependent on it, and so is thus the leading part of the non-dimensional bending moment $(8a^0\,\mathrm{d}w^0/\mathrm{d}z)\,H$.

To verify this, we neglect small effects of surface tension and gravity, and consider instead of (3.1) the following non-dimensional model problem similar to that of Tchavdarov *et al.* (1993) for buckling jets:

$$\frac{\mathrm{d}a^0w^0}{\mathrm{d}z} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}z}\left(a^0\frac{\mathrm{d}w^0}{\mathrm{d}z}\right) = 0, \tag{4.5a,b}$$

$$z = 0: \quad a^0 = 1, \quad w^0 = 1; \qquad z = L: \quad w^0 = E, \tag{4.5c–e}$$

where $E = w_1^0/w_0^0$ is a given parameter.

Equations (4.5a,b) are obtained from (2.22a,b,c). The boundary condition at the plate (4.5e) corresponds to a permeable plate. This condition enables us to mimic the deceleration of the film by the plate in the case when a given velocity of liquid suction into the plate w_1^0 is less then w_0^0 and thus, $E = w_1^0/w_0^0 < 1$. As shown by Tchavdarov *et al.* (1993) this is a realistic first approximation of the unperturbed flow.

The solution of (4.5) is

$$a^0 = E^{-z}, \quad w^0 = E^z. \tag{4.6a,b}$$

The compressive force is thus $8a^0\,\mathrm{d}w^0/\mathrm{d}z = 8\ln E = 8\ln(w_1^0/w_0^0)$. It depends only slightly (logarithmically) on w_0^0 and so does the leading part of the bending moment. Therefore, L_1/h_0 based on the model unperturbed flow depends only slightly on w_0^0 (as well as on the flow rate) for small values of E. The corresponding results for buckling jets are shown in Tchavdarov *et al.* (1993, figure 5).

Similarly, the non-dimensional compressive force and the moment of force based on the unperturbed solution (3.4) and (3.6) for an impermeable plate depend only slightly on the flow rate. The latter manifests itself in the near-independence of the critical height L_1/h_0 of the flow rate. The corresponding results for buckling jets are shown in Tchavdarov *et al.* (1993, figure 12).

Solution of the eigenvalue problem (3.10) and (3.11) enables us to predict the folding frequency ω_*. The non-dimensional folding frequency at the onset of instability,

$$\bar{\omega} = \frac{\omega_* L}{w_0^0} = \mathrm{Im}\{\Psi\}, \tag{4.7a,b}$$

is plotted in figure 5.

The experimental data of Griffiths & Turner (1988) is not used for comparison in the present work for the following reasons. First, their results on impingement of a plane jet on an interface between two liquids are outside the scope of the present consideration, where the free surface of the film is supposed to be free from large stresses; the latter is characteristic of liquid films in air. Second, their results on liquid films impinging upon the free surface of a liquid were mostly obtained with rather thick films, where the length/thickness ratio seems to be too small to apply the quasi-one-dimensional approach. Third, several physical parameters, characterizing their experiment are unknown.

5. Conclusions

Folding instability in liquid films is analogous to buckling instability in impinging liquid jets. It sets in under the action of compressive viscous forces in the film. Under the quasi-one-dimensional approach, the folding height is predicted fairly well only up to a value of $\mu Q'/\rho g h_0^3$ of the order of several units. For higher values of this parameter the theory should be based on the complete set of equations of hydrodynamics and account for the inertial effects.

The nonlinear equations (2.9*a*–*c*) obtained in the present work are valid for large deflections of the film provided inequality (2.1) is not violated. Therefore, (2.9*a*–*c*) form a basis for a future study of a fully nonlinear problem, which may describe the folding phenomenon after instability has set in.

A. L. Y. is a recipient of the Guastalla Fellowship established by Fondation Rashi, Planning and Grants Committee of the Council of Higher Education, the Israel Academy of Sciences and Humanities. This research was partially supported by the Fund for Promotion of Research at the Technion. B. M. T. is a short-term recipient of a European Community fellowship at Oxford University where the present research was partially supported by the EC grant as well as the EC Copernicus programme, reference No. CIPA-CT93-0236.

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